Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-09-15

Conceptual questions

Question 1.

(a) Warmup: Find a function $f(x)$ such that $f(0) = 3$, $f'(0) = 2$, and $f''(0) = 4$.
(b) Now onto MVC: Find a function $f(x, y)$ such that

$$f(0,0) = 3, f_x(0,0) = -2, f_y(0,0) = 0,$$
$$f_{xx}(0,0) = 7, f_{xy}(0,0) = -8, f_{yy}(0,0) = 3.$$

Question 2. Let $f(x, y)$ and $g(u, v)$ be two functions, related by

$$g(u, v) = f(e^u + \sin v, e^u + \cos v).$$

Use the following values to calculate $g_u(0,0)$ and $g_v(0,0)$ (not all of the below values may be relevant!).

$$f(0,0) = 3 \quad g(0,0) = 6 \quad f_x(0,0) = 4 \quad f_y(0,0) = 8$$
$$f(1,2) = 6 \quad g(1,2) = 3 \quad f_x(1,2) = 2 \quad f_y(1,2) = 5$$

Computations

Problem 1. Suppose that S is a (nice, differentiable...) surface which contains the two curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle,$$
$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle.$$

Using this information, compute an equation of the tangent plane to S at the point $P(2, 1, 3)$.

Problem 2. The intersection of the plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse C. Find the tangent line to C at the point $(-1, -1, 2)$.

Problem 3. Oops, this is about stuff we haven't covered! Will be recycled for later...

(a) Suppose that $\mathbf{r}(t) = (x(t), y(t))$ is parametrized by arclength (recall that this means $|\mathbf{r}'(t)| = 1$; the particle is "moving at speed 1"). Show that the directional derivative of f in the direction of $\mathbf{r}'(t)$ is equal to $\frac{d}{dt}(f(\mathbf{r}(t)))$. Hint: Use the chain rule.
(b) Consider the function

$$f(x, y) = \cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right).$$

Using the preceding part, compute $f_y(1, 0)$. Hint: Use the unit circle.

Problem 4. Fix a nonnegative integer (this was revised from the original worksheet, for technical reasons) a and consider the function

$$f(x, y) = \begin{cases} \dfrac{(x + y)^a}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For what choices of a is the function f continuous?

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) By Taylor expansion, or other means, one might arrive at the function

$$f(x) = 3 + 2x + 2x^2.$$

(b) The following function will suffice, as you can check:

$$f(x,y) = 3 - 2x + \frac{7}{2}x^2 - 8xy + \frac{3}{2}y^2 + 87x^{30}.$$

Of course the last term is completely unnecessary; it is just there to remind you that the answer is not unique.

Question 2. Let $x = e^u + \sin v$ and $y = e^u + \cos v$. The important part of this problem is understanding that the partial derivatives of f at $(1,2)$ are the relevant ones, not at $(0,0)$, because when $(u,v) = (0,0)$, $(x,y) = (1,2)$So:

$$g_u(0,0) = f_x(1,2)e^0 + f_y(1,2)e^0 = \boxed{7}$$

$$g_v(0,0) = f_x(1,2)\cos(0) + f_y(1,2)(-\sin 0) = \boxed{2}.$$

Answers to computations

Problem 1. $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 2,1,3\rangle$, so we care about $t = 0$ and $u = 1$. At these parameter values, we have $\mathbf{r}_1'(0) = \langle 3,0,-4\rangle$ and $\mathbf{r}_2'(1) = \langle 2,6,2\rangle$. Use their cross product as a normal vector for the plane:

$$24(x-2) - 14(y-1) + 18(z-3) = 0.$$

Problem 2. The tangent plane to the paraboloid at the point in question is

$$z = 2 - 2(x+1) - 2(y+1) = -2 - 2x - 2y.$$

The tangent line to C at the point is the intersection of this plane with the other plane $x + y + 2z = 2$. This can be parametrized as

$$\mathbf{L}(t) = \langle -1,-1,2\rangle + t\langle -3,3,0\rangle.$$

Problem 4. In other words, when is

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^a}{x^2+y^2} = 0?$$

It turns out that this limit does not exist when $a = 0,1,2$ (testing along lines is sufficient). The limit does exist and is equal to zero when $a \geq 3$; perhaps this is most easily seen with polar coordinates and the Squeeze Theorem.